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Lattice Operations Between Observables in Axiomatic Quantum Mechanics

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Observables are treated as σ-homomorphisms of the Borel sets of the real line into an orthomodular σ-lattice L. By means of corresponding spectral-resolutions operations meet and join are defined between observables which endow the set of all observables with a lattice structure in case L is σ-continuous and which give rise to lattices of observables in case L is chosen arbitrarily and the observables commute.

1. INTRODUCTION AND BASIC DEFINITIONS

Let $\mathbb{B}$ denote the Borel sets of the real line and L be an orthomodular σ-lattice (the logic of the quantum mechanical system). Then an observable is a σ-homomorphism of $\mathbb{B}$ into L (see Varadarajan, 1968). We denote the set of all observables, i.e., the set of all σ-homomorphisms of $\mathbb{B}$ into L, by o. There are several approaches to consider for o or subsets of o as an algebraic structure. The best known among these approaches are the classical concepts of operator-algebras (see Bratelli and Robinson, 1979) and the various descriptions of a sum of two observables (see Dvurečenskij, 1980; Gudder, 1965; Varadarajan, 1968). In this note we propose a way to endow o or subsets of o with a lattice structure, so as to construct observables easily by forming *suprema* and *infima* of other observables. For this purpose let C be the set of all left-open and right-closed intervals of the real line $\mathbb{R}$, including the empty set $\varnothing$:

$$C = \{(-\infty, \mu] \mid \mu \in \mathbb{R}\} \cup \{\varnothing\}$$

We consider mappings $\alpha\colon C \to L$ with the properties

$$\alpha(\varnothing) = 0 \tag{1}$$

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and

$$\alpha\Big(\bigcap_{i\in I}\lambda_i\Big)=\bigcap_{i\in I}\alpha(\lambda_i) \tag{2}$$

$\lambda_i \in C$ and I are countable sets. (Here and in the following the least and the greatest elements of L are denoted by 0 and 1, and $\bigcap$ and $\bigcup$ mean the meet and join of L and $\mathbb{B}$.)

Let H be the collection of all mappings α satisfying (1) and (2). Then clearly the restriction of an observable of $\mathbb{B}$ to C belongs to H. On the other hand, using results of Catlin (1968) and Dvurečenskij (1980) one can verify easily (see Lemma 1).

$\alpha \in H$ can be extended to a σ-homomorphism of $\mathbb{B}$ into L if and only if $\bigcup\alpha(C)$ (i.e., $\bigcup_{\lambda\in C}\alpha(\lambda)$) exists in L and equals to 1. If this is the case the extension of α is unique. It shall be denoted by $\bar{\alpha}$. Thus $o=\{\bar{\alpha}|\alpha\in H, \bigcup\alpha(C)=1\}$—we agree to write α for the restriction of an observable $\bar{\alpha}$ to C.

Now, for $\alpha, \beta \in H$ we introduce $\alpha\cap\beta$ and $\alpha\cup\beta$, as usual by pointwise definition:

$$(\alpha\cap\beta)(\lambda)=\alpha(\lambda)\cap\beta(\lambda) \text{ for } \lambda\in C$$

and

$$(\alpha\cup\beta)(\lambda)=\alpha(\lambda)\cup\beta(\lambda) \text{ for } \lambda\in C$$

This definition suggests that we define the meet and join of observables by

$$\bar{\alpha}\cap\bar{\beta}=\overline{\alpha\cap\beta} \qquad \text{and} \qquad \bar{\alpha}\cup\bar{\beta}=\overline{\alpha\cup\beta}$$

The main result of the note is that if L is σ-continuous or if $\bar{\alpha}$ and $\bar{\beta}$ commute $\bar{\alpha}\cap\bar{\beta}$ and $\bar{\alpha}\cup\bar{\beta}$ are also observables, and from this we will derive the lattice structure of o and of certain subsets of o.

A logic L is called σ-continuous if for any ascending chain $x_1\le x_2\le \cdots$ and $a\in L$ the equation

$$a\cap\bigcup_i x_i=\bigcup_i(a\cap x_i)$$

holds.

For all concepts used here but not defined, see e.g., Varadarajan (1968) and Grätzer (1978).

2. LATTICES OF OBSERVABLES

Lemma 1. A mapping $\alpha\in H$ can be extended to a σ-homomorphism of $\mathbb{B}$ into L if and only if $\bigcup\alpha(C)$ exists in L and equals to 1. If this is the case the extension $\bar{\alpha}$ of α is unique.

Proof. That the restriction α of an observable $\bar{\alpha}$ satisfies the condition of Lemma 1 is obvious. To prove the converse define $\alpha((-\infty,\mu])=e_\mu$ for $\mu\in\mathbb{R}$. Then, as it is easy to see, e_μ is a spectral resolution in the sense of

Catlin (1968), i.e., a mapping $e:\mathbb{R}\to L$ such that (i) $\mu\le\nu\Rightarrow e_\mu\le e_\nu$; (ii) $\bigcap_{\mu\in\mathbb{R}} e_\mu=0$ and $\bigcup_{\mu\in\mathbb{R}} e_\mu=1$; (iii) $\bigcap_{\nu<\mu} e_\mu=e_\nu$. As shown by Catlin, the existence of a Boolean sub-σ-algebra S of L such that $\alpha(C)\subseteq S$ implies that α can be extended uniquely to a σ-homomorphism α_S of $\mathbb{B}$ into L with $\alpha_S(\mathbb{B})\subseteq$ S. Considering another Boolean sub-σ-Algebra T with $\alpha(C)\subseteq T$ and taking into account that $S\cap T$ is also a Boolean sub-σ-algebra of L such that $\alpha(C)\subseteq S\cap T$, the application of Catlin's theorem to S, T and $S\cap T$ shows that $\alpha_S=\alpha_T$. Therefore the existence of an arbitrary sub-σ-algebra S of L such that $\alpha(C)\subseteq S$ is a sufficient condition for the existence of a unique extension of α to a σ-homomorphism of $\mathbb{B}$ into L.

In exactly the same way as in the proof of Theorem 1.1 in Dvurečenskij (1980) one can show that there exists a Boolean sub-σ-algebra B_α of L, which is generated by the set $\{\alpha((-\infty,\mu])|\mu\in\mathbb{R}\}$. ■

Remark 1. Instead of using half-open intervals and conditions (1) and (2) in the definition of α, we could have used open intervals and the duals of the conditions (1) and (2), in which case the dual analogue of Lemma 1 would have been an immediate consequence of a result in Catlin (1968). In respect to standard logics, where it is far easier to form the meet of closed subspaces of Hilbert spaces than their join, the author preferred his point of view, which is also indicated in Catlin (1968).

Define $o_c=\{\alpha|\alpha\in H, \bigcup\alpha(C)=1\}$. As one can see at once the condition $\bigcup\alpha(C)=1$ can be replaced by the condition: $\bigcup_{\lambda_i\to\infty}\alpha(\lambda_i)=1$ for any real sequence (λ_i) with $\lambda_i\to\infty$.

As an immediate consequence of the definition of α we find that the algebraic structure $(H,\cap)$ *is a meet-semilattice* and that $(H,\cap,\cup)$ *is a partial lattice* ($\cup$ is not defined everywhere).

Lemma 2. Let L have the property that for any two ascending chains $a_1\le a_2\le\cdots$ and $b_1\le b_2\le\cdots$ with $\bigcup_i a_i=\bigcup_i b_i=1$ also $\bigcup_i(a_i\cap b_i)=1$. Then with $\alpha,\beta\in o_c$ also $\alpha\cap\beta\in o_c$.

Proof. According to the above remarks one has to show that $\bigcup_{\lambda_i\to\infty}\alpha(\lambda_i)=\bigcup_{\lambda_i\to\infty}\beta(\lambda_i)=1$ yields $\bigcup_{\lambda_i\to\infty}(\alpha\cap\beta)(\lambda_i)=1$; yet this is guaranteed by the assumptions of the lemma.

As a short computation shows σ-continuity in a logic L implies that for two countable chains $\{a_i\}$ and $\{b_i\}$ in L

$$\bigcup_i(a_i\cap b_i)=\left(\bigcup_i a_i\right)\cap\left(\bigcup_i b_i\right)$$

and

$$\bigcap_i(a_i\cup b_i)=\left(\bigcap_i a_i\right)\cup\left(\bigcap_i b_i\right)$$

This will be of use in the proofs of the following theorems.

Theorem 1. Let L be σ-continuous. Then $(o, \cap, \cup)$ is a lattice.

Proof. We have to show that $(o_c, \cap, \cup)$ is a lattice. Since $(o_c, \cap, \cup)$ is a substructure of the partial lattice $(H, \cap, \cup)$ it suffices to prove that with $\alpha, \beta \in o_c$ also $\alpha \cap \beta$ and $\alpha \cup \beta \in o_c$. That $\alpha \cap \beta \in o_c$ follows by Lemma 2, to see that $\alpha \cup \beta \in o_c$ we compute:

$$(\alpha \cup \beta)\left(\bigcap_i \lambda_i\right) = \alpha\left(\bigcap_i \lambda_i\right) \cup \beta\left(\bigcap_i \lambda_i\right) = \left(\bigcap_i \alpha(\lambda_i)\right) \cup \left(\bigcap_i \beta(\lambda_i)\right)$$

$$= \bigcap_i (\alpha(\lambda_i) \cup \beta(\lambda_i)) = \bigcap_i ((\alpha \cup \beta)(\lambda_i)),$$

according to the definition of elements of H and the definition of $\alpha \cup \beta$, and by the σ-continuity of L. Thus condition (2) is satisfied for $\alpha \cup \beta$. Since condition (1) is trivially satisfied $\alpha \cup \beta \in H_c$. Because of

$$\bigcup_{\lambda_i \to \infty} (\alpha \cup \beta)(\lambda_i) = \left(\bigcup_{\lambda_i \to \infty} \alpha(\lambda_i)\right) \cup \left(\bigcup_{\lambda_i \to \infty} \beta(\lambda_i)\right)$$

we further obtain $\alpha \cup \beta \in o_c$. ■

A polynomial over $(H, \cap, \cup)$ is a term in elements of H linked together by $\cap$, $\cup$ and parentheses such that the result is a well defined element of H. E.g., $p(\alpha, \beta, \gamma, \delta) = ((\alpha \cap \beta) \cup \gamma) \cap (\delta \cup \alpha)$ with $\alpha, \beta, \gamma, \delta \in H$ is a polynomial over H (see Grätzer, 1978).

The number of symbols apart $\cap$, $\cup$ and parentheses occurring in a polynomial is called the length of the polynomial. Let $A_c = \{\alpha_j | j \in J\}$ be a subset of o_c. Then we denote by A the set $\{\bar{\alpha}_j | j \in J\}$. If $p(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a polynomial over H with $\alpha_1, \alpha_2, \ldots, \alpha_n \in A_c$ we refer to it as a polynomial over A_c. The term $p(\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_n)$ will be called a polynomial over A.

Theorem 2. Let L be an arbitrary logic and A a set of pairwaise commuting observables; further let $P(A)$ be the set of all polynomials over A. Then all elements of $P(A)$ are observables, each two of them commute, and $(P(A), \cap, \cup)$ is a lattice.

Proof. Since all $\bar{\alpha} \in A$ are pairwaise commuting, there exists a Boolean-sub-σ-algebra S of L such that $\bar{\alpha}(\mathbb{B}) \subseteq S$ for all $\bar{\alpha} \in A$ (see Varadarajan, 1968). Hence by the definition of $\cap$ and $\cup$ and by the theorem of Catlin (1968) mentioned above $\bar{\beta}(\mathbb{B}) \subseteq S$ for all $\bar{\beta} \in P(A) \cap o$. From this we can conclude (Varadarajan, 1968) that the elements $\bar{\beta} \in P(A) \cap o$ are pairwise commuting.

Now we show by induction on the length of the polynomials of $P(A_c)$ that $P(A_c) \subseteq o_c$ where from we obtain that $P(A) \subseteq o$. Polynomials over A_c of length 1 are just the elements of A_c.—Now let us assume that for each polynomial of length $m \leq n-1$ our assertion is true and let $p(\alpha_1, \ldots, \alpha_k)$ be a polynomial over A_c of length n. Then either $p = p_1 \cup p_2$ or $p = p_1 \cap p_2$ with p_1, p_2 each being polynomials of length $\leq n-1$, hence $p_1, p_2 \in o_c$. $\overline{p_1}(\mathbb{B}) \subseteq S$ and $\overline{p_2}(\mathbb{B}) \subseteq S$. Since S is a Boolean sub-σ-algebra of L any two elements of S commute, where form it follows that S is σ-continuous (Varadarajan, 1968). Applying Theorem 1 to σ-homomorphisms of $\mathbb{B}$ into S we obtain that $p_1 \cup p_2$ and $p_1 \cap p_2$ are σ-homomorphisms of $\mathbb{B}$ into S, hence in either case $p \in o_c$. That $(P(A), \cap, \cup)$ is a lattice is a consequence of the fact that $(P(A_c), \cap, \cup)$ is a substructure of $(H, \cap, \cup)$. ∎

Remark 2. Dealing with commuting observables one can show that it suffices to make sure that they commute on C, that is to say: if $\alpha(\lambda)$ commutes with $\beta(\mu)$ for all $\lambda, \mu \in C$, then $\bar{\alpha}$ and $\bar{\beta}$ commute.

The definition of meet and join of two observables $\bar{\alpha}, \bar{\beta}$ makes it very easy to determine $\bar{\alpha} \cup \bar{\beta}$ and $\bar{\alpha} \cap \bar{\beta}$ in case the "spectral-resolutions" α and β of $\bar{\alpha}$ and $\bar{\beta}$ are known. If this is not the case, e.g., if the observables in question are discrete and they are given by their values at one-element subsets of $\mathbb{R}$, then it is also not complicated to find $\bar{\alpha} \cap \bar{\beta}$ and $\bar{\alpha} \cup \bar{\beta}$: If, for example $\bar{\alpha}$ and $\bar{\beta}$ commute, then one can show easily that for $\lambda \in \mathbb{R}$:

$$(\alpha \cap \beta)(\{\lambda\}) = (\bar{\alpha}(\{\lambda\}) \cup \bar{\beta}(\{\lambda\})) \cap \left(\bigcup_{\lambda_i \leq \lambda} \bar{\alpha}(\{\lambda_i\}) \right) \cap \left(\bigcup_{\lambda_i \leq \lambda} \bar{\beta}(\{\lambda_i\}) \right)$$

and

$$(\alpha \cup \beta)(\{\lambda\}) = (\bar{\alpha}(\{\lambda\}) \cup \bar{\beta}(\{\lambda\})) \cap \left(\bigcup_{\lambda_i \geq \lambda} \bar{\alpha}(\{\lambda_i\}) \right) \cap \left(\bigcup_{\lambda_i \geq \lambda} \bar{\beta}(\{\lambda_i\}) \right)$$

where $\{\lambda_i\}$ are the one-element subsets of $\mathbb{R}$ with $\alpha(\{\lambda_i\})$ or $\beta(\{\lambda_i\}) \neq 0$. Similar formulas hold for noncommuting discrete variables if L is σ-continuous.

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